

# Massless scalar particle on AdS spacetime: hamiltonian reduction and quantization

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ABSTRACT: We investigate the massless scalar particle dynamics on  $AdS_{N+1}$  ( $N > 1$ ) by the method of Hamiltonian reduction. Using the dynamical integrals of the conformal symmetry we construct the physical phase space of the system as a  $SO(2, N + 1)$  orbit in the space of symmetry generators. The symmetry generators themselves are represented in terms of  $(N + 1)$ -dimensional oscillator variables. The physical phase space establishes a correspondence between the  $AdS_{N+1}$  null-geodesics and the dynamics at the boundary of  $AdS_{N+2}$ . The quantum theory is described by a UIR of  $SO(2, N + 1)$  obtained at the unitarity bound. This representation contains a pair of UIR's of the isometry subgroup  $SO(2, N)$  with the Casimir number corresponding to the Weyl invariant mass value. The whole discussion includes the globally well-defined realization of the conformal group via the conformal embedding of  $AdS_{N+1}$  in the ESU  $\mathbb{R} \times S^N$ .

KEYWORDS: Space-Time Symmetries, AdS-CFT Correspondence, Gauge Symmetry.

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## 1. Introduction

Starting already in the sixties, there exists an extensive literature on unitary irreducible representations (UIR's) of  $SO(2, N)$  and their use either for the conformal group of Minkowski spaces or for the isometry group of  $AdS_{N+1}$ . In recent times the interest in this subject was renewed in particular by the AdS/CFT correspondence [1]. This paper is the third in a series devoted to the study of UIR's of  $SO(2, N)$  in relation to particle dynamics on  $AdS_{N+1}$ . Our results concern the explicit realization of UIR's  $D_N(\alpha)$  of  $SO(2, N)$ , for generic  $N$ , either in terms of operators acting on holomorphic functions of  $N$  complex variables, obtained via geometric quantization [2], or  $N$ -dimensional oscillator variables [3]. Furthermore our study is singled out by being completely based on particle dynamics and its treatment via Hamilton reduction. Within this framework the present paper is devoted to the study of the peculiarities of massless scalar particles. In the field theoretical treatment these peculiarities have been shown to be related to singleton representations [4–6].

The classical action for a particle in  $AdS_{N+1}$  contains its classical mass  $m$  and the  $AdS_{N+1}$  radius  $R$  as parameters. For generic  $m \neq 0$  the symmetry group is  $SO(2, N)$ , as the isometry group of the space-time. On the classical level the quadratic Casimir  $C = \frac{1}{2}J_{AB}J^{AB}$  is equal to  $m^2R^2$  and the lowest possible particle energy  $\alpha$  in units of  $1/R$  is equal to  $mR$ . Quantization [2] deforms the classical relation  $C = \alpha^2$  to  $C_q = \alpha_q(\alpha_q - N)$ .  $C_q/R^2$  is interpreted as the squared mass of the quantum particle and the connection to the classical mass is lost.<sup>1</sup>

Therefore, up to this point the question which value of the Casimir (mass) in the quantum case corresponds to the massless particle cannot be answered. The only possibility to identify the Casimir (mass) of the massless particle is via its enhanced symmetry related to conformal invariance. To do this, there are at least two possibilities. At first one can take into account information from outside particle dynamics and identify the mass of the

<sup>1</sup>After this comment we will drop the index  $q$  for the quantum versions.

quantum particle with the mass in the classical Klein-Gordon equation. There one has Weyl invariance for  $C = -\frac{N^2-1}{4}$ , corresponding to  $\alpha = \frac{N+1}{2}$ . We will comment this option later in the paper, but our main concern will be connected to a second possibility. As announced, we stay completely within particle dynamics, start with the classical action of a massless particle, note its invariance under the larger conformal group of  $AdS_{N+1}$ , which is  $SO(2, N + 1)$ , and quantize this symmetry group.

Various subtleties we will meet in the following are related to the somewhat exotic causal structure of  $AdS_{N+1}$ . This space-time is not a global hyperbolic one. Null geodesics reach the boundary in finite time. There are conjugate points, in particular all time-like geodesics starting at one point meet again after a time interval  $\pi$  at the corresponding  $AdS$  antipodal point, and related to this, only part of the causal future of a point can be reached by time-like geodesics starting at this point [7, 4]. Although these issues play a role in field theory, both for the massless as well as the massive case via the specification of certain boundary conditions, there seems to be no particular problem for generic masses in particle dynamics, the particle stays within the space-time for all times and the space of particle trajectories is mapped one to one to the space of dynamical integrals [3]. However, the situation becomes more involved for the massless particle, its classical trajectories go from boundary to boundary within a time interval  $\pi$  and to a given set of isometric dynamical integrals belong two trajectories. Special care is needed also for the global realization of the conformal group itself.

## 2. The conformal group of $AdS_{N+1}$

The  $(N + 1)$ -dimensional  $AdS$  space can be realized as the hyperboloid of radius  $R$

$$X_0^2 + X_{0'}^2 - \sum_{n=1}^N X_n^2 = R^2 \tag{2.1}$$

embedded in the  $(N + 2)$ -dimensional flat space  $\mathbb{R}_{N+1}^2$  with coordinates  $X_A$ ,  $A = (0, 0', 1, \dots, N)$  and the metric tensor  $G_{AB} = \text{diag}(+, +, -, \dots, -)$ .<sup>2</sup>

As is well-known, the conformal group of the four-dimensional Minkowski space, due to the fact that the special conformal transformations map certain whole light cones to infinity, is globally not well defined within the original space. To cure this problem one has to enlarge the discussion to an infinite-sheeted covering of the Minkowski space, which can be conformally mapped to an Einstein static universe (ESU) [8]. It is straightforward to adopt the techniques in this construction to our situation.

From general theorems on constant curvature spaces it is known that the conformal group of  $AdS_{N+1}$  for  $N \geq 2$  is  $SO(2, N + 1)$ , the same as for the Minkowski space of the same dimension. The case  $N = 1$  has infinite dimensional conformal symmetry and will not be discussed here. What we need are explicit transformation formulas. To derive them we start with the cone in  $\mathbb{R}_{N+1}^2$

$$Y_0^2 + Y_{0'}^2 - Y_1^2 - \dots - Y_N^2 - Y_{N+1}^2 = 0 \tag{2.2}$$

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<sup>2</sup>More precise  $AdS_{N+1}$  is understood as the universal covering of the hyperboloid. If below we talk about its isometry group, we have in mind the related universal covering of  $SO(2, N)$ .

without its vertex  $Y = 0$ . It is invariant under  $SO(2, N + 1)$  transformations

$$Y^{\hat{A}} \mapsto \Lambda^{\hat{A}}_{\hat{B}} Y^{\hat{B}}, \quad \hat{A}, \hat{B} = 0, 0', 1, \dots, N, N + 1. \quad (2.3)$$

We now relate the cone (2.2) to the  $AdS$  hyperboloid (2.1) via

$$X^A = R \frac{Y^A}{Y^{N+1}}, \quad Y^{N+1} > 0, \quad A = 0, 0', 1, \dots, N \quad (2.4)$$

and get under a  $SO(2, N + 1)$  transformation

$$X^A \mapsto \frac{\Lambda^A_B X^B + R \Lambda^A_{N+1}}{\Lambda^{N+1}_{N+1} + R^{-1} \Lambda^{N+1}_B X^B}. \quad (2.5)$$

It is straightforward to check that these maps of the hyperboloid are indeed conformal ones.

Since  $\Lambda \in SO(2, N + 1)$ , one has  $1 = (\Lambda^{N+1}_{N+1})^2 - \Lambda^{N+1}_A \Lambda^{N+1}_A$ . Therefore, either  $\Lambda^{N+1}_A = 0, \forall A$  and  $\Lambda^{N+1}_{N+1} = \pm 1$ , or  $\Lambda^{N+1}_A$  is a non-zero vector in the embedding space of the hyperboloid. In the first case the transformation (2.5) is an isometry of the  $AdS$  hyperboloid and globally well defined. In *all* other cases the transformation is singular where the denominator vanishes. This means that isometries are the only globally well defined conformal transformations.

We now map the cone (2.2) to an ESU  $\mathbb{R} \times S^N$ , defined by  $\theta \in \mathbb{R}, (\vec{Z}, Z^{N+1}) \in \mathbb{R}^{N+1}, \vec{Z}^2 + (Z^{N+1})^2 = 1$  via

$$\tan \theta = \frac{Y^{0'}}{Y_0}, \quad \vec{Z} = \frac{\vec{Y}}{\sqrt{Y_0^2 + Y_{0'}^2}}, \quad Z^{N+1} = \frac{Y^{N+1}}{\sqrt{Y_0^2 + Y_{0'}^2}}. \quad (2.6)$$

Due to (2.4) this induces the standard injective map of  $AdS_{N+1}$  to the ESU

$$\tan \theta = \frac{X^{0'}}{X_0}, \quad \vec{Z} = \frac{\vec{X}}{\sqrt{X_0^2 + X_{0'}^2}}, \quad Z^{N+1} = \frac{R}{\sqrt{X_0^2 + X_{0'}^2}}. \quad (2.7)$$

Since in the last formula  $0 < Z^{N+1} \leq 1$ , the image of  $AdS_{N+1}$  is just a half of the ESU. The other half of ESU then can be considered as a copy of the original  $AdS_{N+1}$ . The conformal boundary of  $AdS_{N+1}$  is mapped to  $\mathbb{R}$  times the  $(N - 1)$ -dimensional equator at  $Z^{N+1} = 0$  of the  $N$ -dimensional ESU sphere. If one introduces on the cone (2.2) equivalence classes  $Y^{\hat{A}} \sim \mu Y^{\hat{A}}, \mu > 0$  the ESU is mapped one to one to these equivalence classes. Since the action of  $SO(2, N + 1)$  commutes with this equivalence relation, it is globally well defined on the ESU.

A last comment concerns the differences to the Minkowski case [8]. There its conformal boundary is fixed by  $Y^{0'} + Y^{N+1} = 0$ . Due to this an infinite number of conformal copies of Minkowski space find its place on the ESU. Furthermore, there at least some of the  $SO(2, N + 1)$  transformations, mixing  $Y^{N+1}$  with other coordinates, keep the boundary invariant. They are combinations of isometries and dilatations.

### 3. Classical theory

A part of the classical description for massless and massive particle dynamics is similar. Therefore, we briefly repeat the scheme of [2] for  $m = 0$  and discuss the items specific for the massless case in more detail.

The massless particle dynamics on the hyperboloid (2.1) is described by the action

$$S = - \int d\tau \left[ \frac{\dot{X}^A \dot{X}_A}{2e} + \frac{\mu}{2} (X^A X_A - R^2) \right], \quad (3.1)$$

where  $e$  and  $\mu$  are Lagrange multipliers and  $\tau$  is an evolution parameter. The role of the time coordinate on the hyperboloid (2.1) is played by the polar angle  $\theta$  on the plane  $(X_0, X_{0'})$ :  $X_0 = r \cos \theta$ ,  $X_{0'} = r \sin \theta$ . To have the kinetic term of the space coordinates  $\dot{X}_n \dot{X}_n$  with a positive coefficient we assume  $e > 0$ , and to fix the time direction we choose  $\dot{\theta} > 0$ , which is equivalent to  $X_0 \dot{X}_{0'} - X_{0'} \dot{X}_0 > 0$ .

The action (3.1) is gauge invariant and by the Dirac procedure we find three constraints

$$X^A X_A - R^2 = 0, \quad P_A P^A = 0, \quad P_A X^A = 0. \quad (3.2)$$

The constraint  $P_A P^A = 0$  is of the first class, whereas the two others are of the second class. Therefore, the dimension of the reduced phase space is  $2N$  like in the massive case. We will describe this space in terms of dynamical integrals.

The spacetime isometry group  $SO(2, N)$  provides the conserved quantities

$$J_{AB} = P_A X_B - P_B X_A, \quad (3.3)$$

where  $P_A$  are the canonical momenta  $P_A = -\dot{X}_A/e$ . Since  $\theta$  is the time coordinate,  $J_{00'}$  is associated with the particle energy  $E$  and due to our assumptions it is positive

$$E = P_0 X_{0'} - P_{0'} X_0 = e^{-1} (X_0 \dot{X}_{0'} - X_{0'} \dot{X}_0) > 0. \quad (3.4)$$

The boosts we denote by  $J_{0n} = K_n$ ,  $J_{0'n} = L_n$  and we also use their complex combinations

$$z_n = L_n - iK_n, \quad z_n^* = L_n + iK_n, \quad n = 1, \dots, N. \quad (3.5)$$

For further calculations it is convenient to introduce the following  $SO(N)$  scalars

$$J^2 = \frac{1}{2} J_{kk'} J_{kk'}, \quad \lambda^2 = z_k^* z_k, \quad \rho^2 = \sqrt{z^{*2} z^2}, \quad e^{2i\beta} = \frac{z^2}{\rho^2}, \quad (3.6)$$

where  $z^2 = z_k z_k$ ,  $z^{*2} = z_k^* z_k^*$  and we assume  $0 \leq \beta < \pi$ .

Due to the constraints (3.2) the  $SO(2, N)$  quadratic Casimir number vanishes,  $C = \frac{1}{2} J_{AB} J^{AB} = 0$ , and this condition can be written as

$$E^2 + J^2 = \lambda^2. \quad (3.7)$$

A set of other quadratic relations  $J_{AB} J_{A'B'} = J_{AA'} J_{BB'} - J_{AB'} J_{BA'}$  follows from (3.3) as identities in the variables  $(P, X)$ . Taking  $A = 0$ ,  $B = 0'$ ,  $A' = m$ ,  $B' = n$  ( $m \neq n$ ) and using (3.5) we obtain

$$2iE J_{mn} = z_m^* z_n - z_n^* z_m. \quad (3.8)$$

Its square yields  $4E^2J^2 = \lambda^4 - \rho^4$  and together with (3.7) we find the following relations between the scalar variables

$$E^2 = \frac{1}{2} (\lambda^2 + \rho^2) , \quad J^2 = \frac{1}{2} (\lambda^2 - \rho^2) . \quad (3.9)$$

Eqs. (3.6), (3.8), (3.9) define  $E$  and  $J_{mn}$  as functions of  $(z_n, z_n^*)$  and, therefore,  $(z_n, z_n^*)$  or  $(K_n, L_n)$  are global coordinates on the space of dynamical integrals of the isometry group. These integrals allow to represent the particle trajectories geometrically without solving the dynamical equations. From (3.3) we find  $N$  equations as identities in the variables  $(P, X)$

$$E X_n = K_n X_{0'} - L_n X_0 . \quad (3.10)$$

Since  $E, K_n, L_n$  are constants, Eq. (3.10) defines a 2-dimensional plane in the embedding space  $\mathbb{R}_N^2$ . The intersection of this plane with the hyperbola (2.1) is a particle trajectory. The plane defined by (3.10) goes through the origin of  $\mathbb{R}_N^2$  and the way how it intersects the hyperboloid depends on the relations between the dynamical integrals  $K_n, L_n$  and  $E$ . To describe the character of trajectories (3.10) we parameterize them by the time coordinate  $\theta$

$$X_0 = r(\theta) \cos \theta , \quad X_{0'} = r(\theta) \sin \theta , \quad X_n = \frac{r(\theta)}{E} (K_n \sin \theta - L_n \cos \theta) \quad (3.11)$$

$$\text{with} \quad r(\theta) = \frac{ER}{\rho |\sin(\theta - \beta)|} . \quad (3.12)$$

The function  $r(\theta)$  here is obtained from the relation  $X_n X_n = r^2 - R^2$  and Eqs. (3.6), (3.9).

The singularities of  $r(\theta)$  correspond to the  $AdS$  boundary and therefore for  $\rho = 0$  the massless particle is always at the boundary. From the isometric point of view this would force us to remove  $\rho = 0$  out of the phase space. However, for implementing conformal invariance one anyway has to switch to the ESU whose one half is conformally mapped to the  $AdS$ . Then  $\rho = 0$  has to be kept within the phase space. The corresponding trajectories are completely inside the equator of the ESU.

If  $\rho \neq 0$ , the singularity of (3.12) at  $\theta - \beta = k\pi$  indicates that the massless particle always reaches the  $AdS$  boundary and, for a fixed  $(K_n, L_n)$ , there are two different null-geodesics given in the time intervals  $\theta \in (\beta, \pi + \beta)|_{\text{mod } 2\pi}$  and  $\theta \in (\pi + \beta, 2\pi + \beta)|_{\text{mod } 2\pi}$ , respectively. The pieces of both trajectories, which are disconnected with respect to  $AdS$ , represent the “visible” parts of two smooth trajectories in the full ESU. These are just two with luminal velocity driven great circles on the ESU sphere, which intersect each other on the equator.

To complete the description of the reduced phase space and its relation to the  $AdS$  null-geodesics we introduce additional dynamical integrals related to the conformal symmetry. The action (3.1) is invariant under the conformal transformations, since the conformal factor of the kinetic term can be compensated by the transformations of the Lagrange multiplier  $e$ . Considering infinitesimal transformations (2.5) different from isometries, we find the Killing vectors (the down index labels them)

$$\mathcal{K}_A^B = R \delta_A^B - R^{-1} X_A X^B . \quad (3.13)$$

The corresponding conserved quantities are  $C_A = \mathcal{K}^B{}_A P_B = RP_A - R^{-1}(PX) X_A$  and on the constraint surface (3.2) they become

$$C_A = RP_A. \tag{3.14}$$

The conservation of canonical momenta  $P_A$  in the massless case can also be checked directly from the dynamical equations  $\dot{X}_A = -eP_A$ ,  $\dot{P}_A = \mu X_A$ . Multiplying the first equation by  $P^A$ , the second by  $X^A$  and using the constraints (3.2), we find  $\mu = 0$ , which provides  $\dot{P}_A = 0$ . Due to the conservation of  $P_A$ , the null-geodesics are straight lines in  $\mathbb{R}^2_N$ . Making use of (3.11) and (3.12) we represent the trajectories as  $X_A = C_A T + Q_A$ , where  $T = ER\rho^{-2} \cot(\theta - \beta)$  is a parameter along the lines, and

$$Q_0 = \mp \frac{ER}{\rho} \sin \beta, \quad Q_{0'} = \pm \frac{ER}{\rho} \cos \beta, \quad Q_n = \pm \frac{R}{\rho} (K_n \cos \beta - L_n \sin \beta) \tag{3.15}$$

The two signs above correspond to two lines given for the same set of isometry generators.

Since the number of functionally independent dynamical integrals has to be  $2N$ , we investigate relations between  $J_{AB}$  and  $C_A$ . From (3.3) and (3.14) we find

$$EC_n = K_n C_{0'} - L_n C_0 \quad (n = 1, \dots, N), \tag{3.16}$$

as identities in the variables  $(P, X)$ , like in (3.10). Two other relations

$$K_k L_k + C_0 C_{0'} = 0 \quad \text{and} \quad K_k K_k - L_k L_k + C_0^2 - C_{0'}^2 = 0 \tag{3.17}$$

also follow from (3.3) and (3.14), but on the mass-shell (3.2) only. Eqs. (3.8), (3.9), (3.16) define all generators of the conformal symmetry as functions of  $K_n, L_n, C_0, C_{0'}$  and these  $2N + 2$  dynamical integrals are constrained by (3.17). Introducing the complex variable  $w = C_{0'} - iC_0$ , the two equations of (3.17) can be combined in a one complex relation

$$z^2 + w^2 = 0. \tag{3.18}$$

Because of  $E > 0$ , the vector  $z_n$  is nonzero and, therefore, the space of dynamical integrals defined by (3.18) is a regular  $2N$ -dimensional manifold. This manifold is identified with the reduced phase space, which is the physical phase space  $\Gamma_{ph}$  of the system. The isometry generators  $(K_n, L_n)$  are only local coordinates on  $\Gamma_{ph}$  and by (3.18) we have

$$w = \mp i\sqrt{z^2} = \mp i\rho e^{i\beta}. \tag{3.19}$$

Eq. (3.4) yields  $E = e^{-1}r^2(\theta)\dot{\theta}$  and we can express the Lagrange multiplier  $e$  through the dynamical variables. Then, calculating  $w = Re^{-1}(\dot{X}_{0'} - i\dot{X}_0)$  on the trajectories (3.11)–(3.12) for the two different intervals  $\theta \in (\beta, \pi + \beta)$  and  $\theta \in (\pi + \beta, 2\pi + \beta)$ , we find  $w = -i\rho e^{i\beta}$  and  $w = i\rho e^{i\beta}$ , respectively. Thus, the above mentioned two trajectories correspond to the two possible values of  $w$  in (3.19).

As far as the velocities of the massless particle are constrained by  $\dot{X}_A \dot{X}^A = 0$ , the set of null-geodesics is  $(2N - 1)$ -dimensional and, unlike to the massive case, there is no one to one correspondence between the trajectories and the space of dynamical integrals.

Nevertheless, the set of all trajectories reflects the structure of  $\Gamma_{ph}$ . They are invariant under re-scalings of all dynamical integrals  $J_{AB} \mapsto e^\gamma J_{AB}$ . Hence the set of trajectories can be identified with  $\Gamma_{ph}/\mathbb{R}_+$ .

Now we describe the Poisson bracket structure of  $\Gamma_{ph}$ . The  $so(2, N)$  Poisson bracket algebra of the isometry generators (3.3) is obviously preserved after the reduction to  $\Gamma_{ph}$ . It can be written in the form

$$\{z_m^*, z_n\} = 2J_{mn} - 2i\delta_{mn} E, \quad \{z_m, z_n\} = 0 = \{z_m^*, z_n^*\}, \quad (3.20)$$

$$\{J_{lm}, z_n\} = z_l \delta_{mn} - z_m \delta_{ln}, \quad \{E, z_n\} = -iz_n, \quad \{E, J_{mn}\} = 0, \quad (3.21)$$

and the  $J_{mn}$ 's form a  $so(N)$  subalgebra. Since the conformal generators  $C_A$  do not commute with the second class constraints of (3.2), their Poisson brackets are deformed after the Hamiltonian reduction to  $\Gamma_{ph}$ . To calculate such reduced brackets we use that

$$\{z_m^*, z^2\} = \frac{2}{iE} (\rho^2 z_m + z^2 z_m^*), \quad (3.22)$$

which follow from (3.20) due to (3.9). Then, writing (3.16) in the form  $2iEC_n = z_n^* w - z_n w^*$ , from (3.18) and (3.22) we find

$$\{z_n^*, w\} = 2C_n, \quad \{w, w^*\} = 2i E. \quad \{z_n, w\} = 0. \quad (3.23)$$

These equations define the Poisson brackets between other conformal generators and the result can be written in the form

$$\{J_{AA'}, C_B\} = G_{AB} C_{A'} - G_{A'B} C_A, \quad \{C_A, C_B\} = -J_{AB}. \quad (3.24)$$

The Poisson brackets (3.24) extend the  $so(2, N)$  algebra (3.20)–(3.21) up to  $so(2, N+1)$ , which describes the underlying conformal symmetry. Adding one column and one row to the antisymmetric matrix  $J_{AB}$  by the scheme

$$J_{\hat{A}\hat{B}} = \begin{pmatrix} J_{AB} & C_A \\ -C_A & 0 \end{pmatrix}, \quad \hat{A}, \hat{B} = (0, 0', 1, \dots, N+1) \quad (3.25)$$

we get  $(N+3) \times (N+3)$  antisymmetric  $J_{\hat{A}\hat{B}}$  and the Poisson brackets of its components correspond to the  $so(2, N+1)$  algebra in the standard covariant form.

We specify the compact and non-compact generators for the  $SO(2, N+1)$  symmetry by  $E$ ,  $J_{\hat{m}\hat{n}}$  and  $z_{\hat{n}}$ ,  $z_{\hat{n}}^*$  ( $\hat{m}, \hat{n} = 1, \dots, N+1$ ), respectively. So, we use the same notations as for  $SO(2, N)$ , only the indices run from 1 to  $N+1$ . To distinguish the  $SO(N+1)$  scalars we use the sign ‘hat’

$$\hat{J}^2 = \frac{1}{2} J_{\hat{k}\hat{k}'} J_{\hat{k}\hat{k}'}, \quad \hat{\lambda}^2 = z_{\hat{k}}^* z_{\hat{k}}, \quad \hat{z}^2 = z_{\hat{k}} z_{\hat{k}}, \quad \hat{z}^{*2} = z_{\hat{k}}^* z_{\hat{k}}^*. \quad (3.26)$$

Due to the mass-shell condition  $P_A P^A = 0$  and the relation  $C_A = R P_A$ , the Casimir number for  $J_{\hat{A}\hat{B}}$  is also zero. In terms of  $SO(N+1)$  scalars this condition reads

$$E^2 + \hat{J}^2 - \hat{\lambda}^2 = 0. \quad (3.27)$$



Eq. (3.16) provides that the quadratic relations (3.8) are fulfilled for the components  $J_{nN+1}$  as well. As a result

$$2iE J_{\hat{m}\hat{n}} = z_{\hat{m}}^* z_{\hat{n}} - z_{\hat{n}}^* z_{\hat{m}} \tag{3.28}$$

is valid for all indices  $\hat{m}$  and  $\hat{n}$ . Finally, the condition (3.18) becomes

$$\hat{z}^2 = 0. \tag{3.29}$$

Thus, the physical phase space of the massless particle  $\Gamma_{ph}$  is identified with the space of  $SO(2, N + 1)$  generators  $E, J_{\hat{m}\hat{n}}, z_{\hat{n}}, z_{\hat{n}}^*$ , which satisfy the Eqs. (3.27)–(3.29) with  $E > 0$  and non-zero vector  $z_{\hat{n}}$ . Note that, due to (3.22) and the trivial Poisson brackets  $\{z_m, z^2\} = 0, \{J_{mn}, z^2\} = 0, \{E, z^2\} = -2iz^2$ , the condition  $z^2 = 0$  is  $SO(2, N)$  invariant and, therefore, the manifold  $\hat{z}^2 = 0$  is  $SO(2, N + 1)$  invariant.

#### 4. Quantum theory

A consistent quantum theory of the massless particle should provide a realization of the  $SO(2, N + 1)$  symmetry based of the classical picture. The Poisson bracket relations of the  $so(2, N + 1)$  algebra (3.24) are essentially non-linear in terms of the independent variables and their direct representation seems more complicated than in the massive case, since one has to realize a higher symmetry on a non-trivial phase space of the same dimensionality.

We realize the  $SO(2, N + 1)$  symmetry by representations  $D_{N+1}(\alpha)$  [3], which are based on the creation-annihilation operators  $(a_{\hat{m}}^*, a_{\hat{m}})$  of a  $(N + 1)$ -dimensional oscillator. The generators of  $SO(N + 1)$  rotations in  $D_{N+1}(\alpha)$  have the standard quadratic form

$$J_{\hat{m}\hat{n}} = i(a_{\hat{n}}^* a_{\hat{m}} - a_{\hat{m}}^* a_{\hat{n}}), \tag{4.1}$$

and the operator  $\hat{J}^2$  in terms of the creation-annihilation operators, respectively, is

$$\hat{J}^2 = \frac{1}{2} J_{\hat{m}\hat{n}} J_{\hat{m}\hat{n}} = \hat{H}^2 + (N - 1)\hat{H} - \hat{a}^{*2} \hat{a}^2. \tag{4.2}$$

Here  $\hat{H} = a_{\hat{n}}^* a_{\hat{n}}$  is the normal ordered  $(N + 1)$ -dimensional oscillator Hamiltonian,  $\hat{a}^2 = a_{\hat{n}} a_{\hat{n}}$  and  $\hat{a}^{*2} = a_{\hat{n}}^* a_{\hat{n}}^*$ .

The energy operator  $E$  is given as a shifted oscillator Hamiltonian

$$E = \hat{H} + \alpha, \tag{4.3}$$

and  $\alpha$  coincides with its minimal eigenvalue. The compact subalgebra  $so(2) \times so(N + 1)$ , thus, is realized automatically.

The operators  $z_{\hat{n}}$  and  $z_{\hat{n}}^*$  are represented by

$$z_{\hat{n}} = \frac{1}{\sqrt{2\hat{H} + 4\alpha - N + 1 + 2\hat{F}}} \left( (2\hat{H} + 2\alpha + \hat{F}) a_{\hat{n}} - a_{\hat{n}}^* \hat{a}^2 \right), \tag{4.4}$$

$$z_{\hat{n}}^* = \left( a_{\hat{n}}^* (2\hat{H} + 2\alpha + \hat{F}) - \hat{a}^{*2} a_{\hat{n}} \right) \frac{1}{\sqrt{2\hat{H} + 4\alpha - N + 1 + 2\hat{F}}}, \tag{4.5}$$

where  $\hat{F}$  is the following real function of scalar operators

$$\hat{F} = \sqrt{\hat{a}^{*2}\hat{a}^2 + 2(2\alpha - N + 1)(\hat{H} + \alpha)}. \quad (4.6)$$

By (4.2)  $\hat{F}$  can be written as a function of  $\hat{H}$  and  $\hat{J}^2$  and it corresponds to  $F_{N+1}$  of [3]. The operator expressions with square roots in (4.4)–(4.6) depend on commuting operators and there is no ordering problem inside such expressions. They are naturally defined on a basis of mutual eigenstates of  $\hat{H}$ ,  $\hat{J}^2$  and Cartan generators of the  $SO(N + 1)$  rotations as multiplication operators. The form of the boost operators (4.4) guarantees the correct commutation relations between the compact and noncompact generators. The commutators of the operators (4.4) can be calculated by the exchange relations between the creation-annihilation and scalar operators and these calculations complete the commutation relations of the  $so(2, N + 1)$  algebra.

The representations  $D_{N+1}(\alpha)$  are unitary and irreducible if  $\alpha$  is above the unitarity bound  $\alpha > \frac{N-1}{2}$ . The analysis of irreducibility of  $D_{N+1}(\alpha)$  uses the relation between the scalar operator  $\hat{z}^2$  and  $\hat{a}^2$  [3]

$$\hat{z}^2 = \hat{F} \hat{a}^2. \quad (4.7)$$

On the basis of this relation we introduce the quantum analog of (3.29) by

$$\hat{a}^2 |\psi\rangle_{ph} = 0. \quad (4.8)$$

The Hilbert subspace  $\mathcal{H}_{ph}$  defined by this condition can be  $SO(2, N + 1)$  invariant only if  $\alpha$  is just at this bound  $\alpha = \frac{N-1}{2}$ , where the representation is still unitary, but becomes reducible. At the unitarity bound the operator (4.6) reduces to  $\sqrt{\hat{a}^{*2}\hat{a}^2}$  and it vanishes on the solutions of (4.8). Then, the operators (4.4) and (4.5) on  $\mathcal{H}_{ph}$  become

$$z_{\hat{n}} |\psi\rangle_{ph} = \sqrt{2\hat{H} + N - 1} a_{\hat{n}} |\psi\rangle_{ph}, \quad (4.9)$$

$$z_{\hat{n}}^* |\psi\rangle_{ph} = \left( a_{\hat{n}}^* \sqrt{2\hat{H} + N - 1} - \hat{a}^{*2} a_{\hat{n}} \frac{1}{\sqrt{2\hat{H} + N - 1}} \right) |\psi\rangle_{ph}. \quad (4.10)$$

The invariance of  $\mathcal{H}_{ph}$  with respect to the subset of infinitesimal transformations generated by  $E$  and  $J_{\hat{m}\hat{n}}$  is apparent. For the boosts  $z_{\hat{n}}$ ,  $z_{\hat{n}}^*$  one has to use the commutation relations  $[\hat{a}^2, \hat{H}] = 2\hat{a}^2$ ,  $[\hat{a}^2, \hat{a}^{*2}] = 4\hat{H} + 2N + 2$ . Then, for example, from (4.10) we obtain

$$\hat{a}^2 z_{\hat{n}}^* |\psi\rangle_{ph} = a_{\hat{n}}^* \sqrt{2\hat{H} + N + 3} \hat{a}^2 |\psi\rangle_{ph} = 0, \quad (4.11)$$

and altogether find that  $\mathcal{H}_{ph}$  is  $SO(2, N + 1)$  invariant indeed.

The adjoint form of (4.7) is  $\hat{z}^{*2} = \hat{a}^{*2} \hat{F}$  and, therefore,  $\hat{z}^{*2}$  vanishes on  $\mathcal{H}_{ph}$

$$z_{\hat{n}}^* z_{\hat{n}}^* |\psi\rangle_{ph} = 0. \quad (4.12)$$

The obtained representations of the conformal symmetry we denote by  $\mathcal{C}_N$  and now we investigate its structure in more detail. First note that according to (4.10) the vacuum is not invariant under the action of the operators  $z_{\hat{n}}^*$  and the physical states are obtained

by multiple actions of  $z_n^*$ 's on the vacuum state  $|\psi\rangle_{ph} = (z_1^*)^{n_1} (z_2^*)^{n_2} \dots (z_{N+1}^*)^{n_{N+1}} |0\rangle$ . To analyze the embedding of the isometry subgroup  $SO(2, N)$  in  $\mathcal{C}_N$  we calculate the operator for the Casimir number of  $SO(2, N)$ . From (4.1), (4.3), (4.9), (4.10) and (4.12) we obtain

$$\left( E^2 + \frac{1}{2} J_{mn} J_{mn} - \frac{1}{2} (z_n^* z_n + z_n z_n^*) \right) |\psi\rangle_{ph} = \frac{1 - N^2}{4} |\psi\rangle_{ph}, \quad (4.13)$$

where the index summation goes from 1 to  $N$ . Eq. (4.13) indicates that  $\mathcal{C}_N$  contains the UIR's of  $SO(2, N)$  corresponding to the Weyl invariant mass value only. Acting on the vacuum state by the operators  $z_n^*$  ( $n = 1, \dots, N$ ) we create a  $SO(2, N)$  invariant subspace  $\mathcal{H}_- \subset \mathcal{H}_{ph}$ . Since the Casimir number and the lowest value of  $E$  of this representation are fixed by  $\frac{1-N^2}{4}$  and  $\frac{N-1}{2}$ , respectively, we get the representation which is unitary equivalent to  $D_N(\frac{N-1}{2})$ . Similarly, acting by the operators  $z_n^*$  ( $n = 1, \dots, N$ ) on the state  $z_{N+1}^* |0\rangle$ , we generate another subspace  $\mathcal{H}_+$  for a new UIR of  $SO(2, N)$ , which is unitary equivalent to  $D_N(\frac{N+1}{2})$ , since now the lowest eigenvalue of  $E$  is  $\frac{N+1}{2}$ .

The direct sum  $\mathcal{H}_- \oplus \mathcal{H}_+$  is  $SO(2, N + 1)$  invariant and we even have

$$\mathcal{H}_{ph} = \mathcal{H}_- \oplus \mathcal{H}_+. \quad (4.14)$$

To see first the  $SO(2, N + 1)$  invariance of  $\mathcal{H}_- \oplus \mathcal{H}_+$ , it is enough to consider the action of the operator  $z_{N+1}^*$  only. This operator naturally maps  $\mathcal{H}_-$  to  $\mathcal{H}_+$ , since it commutes with all  $z_n^*$ . Due to (4.12)  $z_{N+1}^* z_{N+1}^* |\psi\rangle_{ph} = -z_n^* z_n^* |\psi\rangle_{ph}$ , which implies that  $z_{N+1}^*$  also maps  $\mathcal{H}_+$  to  $\mathcal{H}_-$ . To prove that  $\mathcal{H}_- \oplus \mathcal{H}_+$  covers all  $\mathcal{H}_{ph}$ , we introduce nonphysical states obtained by multiple actions of the operator  $\hat{a}^{*2}$  on the physical states from  $\mathcal{H}_- \oplus \mathcal{H}_+$ . The first two levels of the  $(N + 1)$ -dimensional oscillator Fock space contain only the physical states  $|0\rangle$  and  $z_n^* |0\rangle$ . The total number of states on the level  $k$  is given by  $A_{N+1}^k = \frac{(N+1) \dots (N+k)}{k!}$ . These numbers obey

$$A_{N+1}^k = \left( A_N^k + A_N^{k-1} \right) + A_{N+1}^{k-2}. \quad (4.15)$$

$\left( A_N^k + A_N^{k-1} \right)$  here corresponds to the number of physical states on the level  $k$  from  $\mathcal{H}_- \oplus \mathcal{H}_+$  and the rest  $A_{N+1}^{k-2}$  states are obtained by the action of  $\hat{a}^{*2}$  on all states of the level  $(k - 2)$ . These nonphysical states are obviously orthogonal to the physical states. Thus, the physical and nonphysical states constructed in these way cover the full Fock space.

## 5. Conclusions

We investigated the dynamics of scalar massless particles on  $AdS_{N+1}$  using the conformal symmetry generated by the  $so(2, N + 1)$  algebra. To have invariance under a globally well-defined conformal group,  $AdS_{N+1}$  has to be mapped to half of an ESU  $\mathbb{R} \times S^N$  and extended to the full ESU. Then, what concerns conformal invariant dynamics, all trajectories are either completely within the boundary of  $AdS_{N+1}$ , an equator of the  $S^N$ , or have a smooth continuation into the other half of the ESU.

Hamiltonian reduction leads to a  $2N$  dimensional physical phase space  $\Gamma_{ph}$ , which is a  $SO(2, N + 1)$  orbit in the space of generators of the conformal symmetry and the set of trajectories on the ESU is mapped one to one to  $\Gamma_{ph}/R_+$ .

The conformal group  $SO(2, N + 1)$  can also be considered as the isometry group of  $AdS_{N+2}$ , i.e. one dimension higher. Then  $\Gamma_{ph}$  given by the constraint (3.29) describes the kinematical domain corresponding to particle trajectories on the boundary of  $AdS_{N+2}$ . This boundary is conformally equivalent to an ESU  $\mathbb{R} \times S^N$ , i.e. the conformal completion of the original  $AdS_{N+1}$ .

Quantizing the  $SO(2, N + 1)$  symmetry we introduced the quantum analog of the constraint (3.29) by (4.8). Working now with UIR's  $D_{N+1}(\alpha)$  of  $SO(2, N + 1)$ , to guarantee the invariance of the constraint, one first has to choose  $\alpha$  such that the representation becomes reducible and second to constrain oneself to the irreducible component defined by the constraint. This fixes the lowest possible energy of the conformally invariant particle to the unitarity bound of  $SO(2, N + 1)$ , i.e.  $\alpha = \frac{N-1}{2}$ . Analyzing the transformation properties with respect to the isometry subgroup  $SO(2, N)$  of the conformal group  $SO(2, N + 1)$ , we found (indicating by  $\sim$  the restriction to  $AdS_{N+1}$  isometries)

$$\left[ D_{N+1} \left( \alpha = \frac{N-1}{2} \right) \right]_{\text{constrained}} \sim D_N \left( \alpha = \frac{N-1}{2} \right) \oplus D_N \left( \alpha = \frac{N+1}{2} \right). \quad (5.1)$$

This relation is well-known from field theoretical considerations [5] or general representation theory [11]. The l.h.s. of (5.1) is equal to a singleton (Rac) [4–6] and the localization of singletons on the boundary has appeared in field theoretical terms at various places, see e.g. [12].

What we claim to be a new result of this paper is the derivation completely in terms of particle dynamics. On a more detailed level we should add: particle dynamics formulated via Hamiltonian reduction in terms of  $(N + 1)$ -dimensional oscillator variables.

Concerning the isometry group of our original  $AdS_{N+1}$  the representations on the r.h.s. of (5.1) have no special properties in comparison to their relatives at generic values of  $\alpha > \frac{N-2}{2}$ . As mentioned in the introduction, their relevance for the massless quantum particle could have been borrowed from field theory. But then we still would not understand their necessary combination in the form of a direct sum nor the particles transformation properties under the larger conformal group.

There is a very interesting physical interpretation of (5.1): In the generic case  $D_N(\alpha)$  describes a massive (isometric) particle which lives inside  $AdS_{N+1}$ , its trajectories never reach the  $AdS$ -boundary.  $D_N(\frac{N\pm 1}{2})$  still share the property that the particle lives in one half of the related ESU. Its classical trajectories are now null geodesics reaching the boundary and being continued by reflection. On the other side, the massless (conformal) particle has to live in the whole ESU. Therefore, it is just the quantum mechanical superposition of the isometric particle with  $\alpha = \frac{N\pm 1}{2}$  living in one or the other half of the ESU.

A last comment concerns related issues in field theory [9]. There, to handle the problems posed by the lack of global hyperbolicity, for the Weyl invariant situation three possible quantization schemes related to transparent and reflective Neumann or Dirichlet boundary conditions have been considered. Obviously our conformal particle corresponds to the transparent case, while the both reflective versions are related to the use of a single  $D_N(\frac{N\pm 1}{2})$ . That the reflective versions violate conformal invariance can also be seen directly from the propagators. They are singular at light cones centered at  $AdS$  antipodal

points [9, 10], but being antipodal is not preserved under general  $SO(2, N + 1)$  transformations (2.5). However, note that this objection applies only if one insists on connecting the notion of masslessness also in field theory with invariance under a globally well-defined conformal group.

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